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SOME SPECIAL FEATURES OF THERMOCONVECTIVE MOTION IN MULTILAYER LIQUIDS

V. M. Shevtsova and A. E. Indeikina

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Thermocapillary convection is considered in multilayer systems of liquids at whose interfaces surface tension forces act. It is assumed that the liquids are immiscible and a linear temperature distribution is maintained at the lower solid boundary.

At present, in modeling convective phenomena one-layer liquids are usually considered. Thermal convection laws in multilayer liquid systems are of interest for understanding processes of fluid dynamics and heat and mass transfer when creating new materials.

We consider motion in a system consisting of N flat layers of viscous incompressible liquids with a thickness D_i, $i = 1, N$, confined between solid surfaces. The index $i = 1$ pertains to the lower layer. It is assumed that the liquids are immiscible and their density decreases with the layer number: $\rho_{i+1} < \rho_i$, $i = 1, N$. At the lower solid boundary, a constant linear temperature distribution is maintained. The heat conduction of the liquids causes a nonuniform temperature distribution at the interfaces. As a result, tangential thermocapillary stresses develop, thus causing motion in the system. It is assumed that all surface tension coefficients depend on the temperature in the following fashion [1]:

$$\sigma_i = \sigma_{0i} + 1/2\alpha_i (T_i - T_0)^2, \quad \sigma_{0i} = \text{const}, \ \alpha_i = \text{const}, \ i = 1, \ N-1.$$

Here T_0 is the temperature corresponding to the extremal surface tension coefficient. Steady-state motion is considered in a multilayer liquid. The geometry of the problem for a three-layer liquid system is shown in Fig. 1.

With ordinary simplifying assumptions, the problem may be mathematically represented by Navier-Stokes, heat conduction, and continuity equations:

$$\begin{gathered} U_i \frac{\partial U_i}{\partial X} + v_i \frac{\partial U_i}{\partial Y} = -\frac{1}{\rho_i} \frac{\partial P_i}{\partial X} + \nu_i \nabla^2 U_i, \\ U_i \frac{\partial v_i}{\partial X} + v_i \frac{\partial v_i}{\partial Y} = -\frac{1}{\rho_i} \frac{\partial P_i}{\partial Y} + \nu_i \nabla^2 v_i, \\ \frac{\partial U_i}{\partial X} + \frac{\partial v_i}{\partial Y} = 0, \quad U_i \frac{\partial T_i}{\partial X} + v_i \frac{\partial T_i}{\partial Y} = \chi_i \nabla^2 T_i, \ i = 1, \ N. \end{gathered} \tag{1}$$

At the solid boundaries the conditions of liquid adhesion are prescribed, at the lower boundary a constant linear temperature distribution is maintained, and the upper boundary is heat insulated:

$$U_1 = v_1 = 0, \quad T_1 = T_0 + AX, \quad Y = 0;$$

$$U_N = v_N = 0, \quad \partial T_N/\partial Y = 0, \quad Y = H.$$

At the liquid interfaces $Y = H_i = \sum_{k=1}^{i} D_k$ we prescribe:

temperature and velocity continuity

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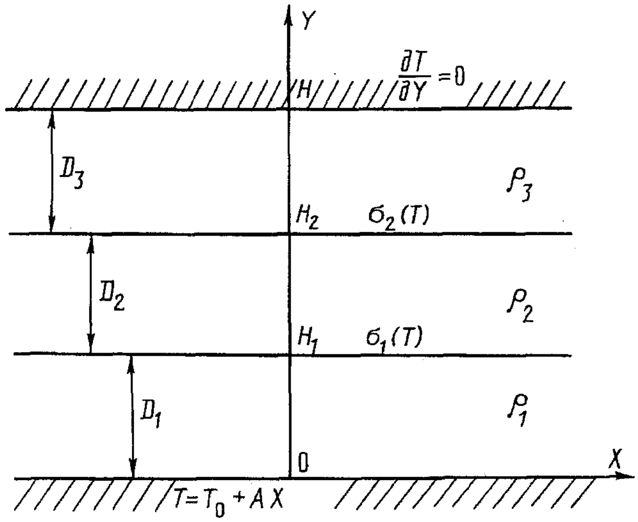

Fig. 1. Geometry of the problem for a three-layer liquid system.

$$U_i = U_{i+1}, \quad T_i = T_{i+1};$$

impermeability condition

$$v_i = v_{i+1} = 0;$$

heat flux continuity

$$k_i \frac{\partial T_i}{\partial Y} = k_{i+1} \frac{\partial T_{i+1}}{\partial Y};$$

balance of viscous forces

$$\eta_i \frac{\partial U_i}{\partial Y} = \eta_{i+1} \frac{\partial U_{i+1}}{\partial Y} + \frac{d\sigma_i}{dT_i} \frac{\partial T_i}{\partial X}.$$

As shown in [2, 3], the assumption of a flat surface is approximately fulfilled for heavy liquids and at a sufficiently high thermocapillary pressure, i.e., at large σ_{0i}.

We introduce dimensionless parameters and variables

$$x = \frac{X}{H}, \quad y = \frac{Y}{H}, \quad d_i = \frac{D_i}{H}, \quad h_i = \frac{H_i}{H},$$

$Pr_i = \nu_i/\chi_i$ is the Prandtl number. We seek a self-similar solution to the problem in the form

$$U_i = \frac{\nu_i}{H} x\psi_i'(y), \quad v_i = -\frac{\nu_i}{H} \psi_i(y), \quad T_i = T_0 + AHx\theta_i(y),$$
$$P_i = P_{0i} - \frac{1}{2} \frac{\rho_i \nu_i^2}{H^2} (\lambda_i x^2 + f_i(y)), \tag{2}$$

where $P_{01} = P(0, 0) = \text{const}$ and $P_{0i} = P(0, h_i) = \text{const}$, $i = 2, N$.

A self-similar solution to the problem on motion in a single liquid layer is found in [3, 4]. To determine new unknown functions $\psi_i(y)$, $\theta_i(y)$, $f_i(y)$ and constant λ_i, we obtain from (1)-(2) the following two-point boundary-value problem for a nonlinear system of ordinary differential equations:

$$\psi_i''' + \psi_i\psi_i'' - (\psi_i')^2 + \lambda_i = 0, \quad f_i' = 2(\psi_i'' + \psi_i\psi_i'),$$
$$\theta_i' - \mathrm{Pr}_i(\psi_i'\theta_i - \psi_i\theta_i') = 0 \tag{3}$$

with the following conditions at the bottom y = 0:

$$\psi_1(0) = \psi_1'(0) = f_1(0) = 0, \quad \theta_1(0) = 1.$$

At the upper boundary y = 1:

$$\psi_N(1) = \psi_N'(1) = f_N(1) = \theta_N'(1) = 0; \tag{4}$$

at the interfaces $y = h_i$; i = 1, N-1:

$$\theta_i(h_i) = \theta_{i+1}(h_i), \quad \theta_i'(h_i) = k_{i+1i}\theta_{i+1}'(h_i), \quad f_i(h_i) = 0,$$
$$\psi_i(h_i) = \psi_{i+1}(h_i) = 0, \quad \psi_i'(h_i) = \nu_{i+1i}\psi_{i+1}'(h_i),$$
$$\psi_i''(h_i) - \eta_{i+1i}\nu_{i+1i}\psi_{i+1}''(h_i) = m_i\theta_i^2(h_i).$$

Here and henceforth $\nu_{ik} = \nu_i/\nu_k$, $\eta_{ik} = \eta_i/\eta_k$, $k_{ik} = k_i/k_k$, $m_i = \alpha_i A^2 H^3(\eta_i \nu_i)$ is the Marangoni number at the i-th interface of liquids.

In order to elucidate characteristic special features of a thermocapillary flow, we obtain an approximate analytical solution for each layer at small Marangoni numbers m_i assuming that all Prandtl numbers Pr_i are of order unity.

There are N-1 Marangoni numbers in the problem, and we may perform expansion in terms of any of them. For definiteness, we shall employ m_1 assuming that all Marangoni numbers are of the same order.

When $m_i = 0$, i = 1, N-1, the problem has the solution $\psi_i = f_i = \lambda_i = 0$, $\theta_i = 1$, which corresponds to a fluid at rest with a uniform temperature distribution with respect to height.

When $|m_i| < 1$, the solution will be constructed by the perturbation method in the form

$$\psi_i = m_1\psi_i^{(1)} + m_1^2\psi_i^{(2)} + \dots, \quad f_i = m_1 f_i^{(1)} + m_1^2 f_i^{(2)} + \dots,$$
$$\lambda_i = m_1\lambda_i^{(1)} + m_1^2\lambda_i^{(2)} + \dots, \quad \theta_i = 1 + m_1\theta_i^{(1)} + m_1^2\theta_i^{(2)} + \dots, \quad i = 1, N. \tag{5}$$

Substituting (5) into (3), (4) and neglecting quadratic terms in m_i, we arrive at

$$\psi_i''' + \lambda_i = 0, \quad f_i' = 2\psi_i'', \quad \theta_i'' = Pr_i\,\psi_i'. \tag{6}$$

To avoid confusion with the layer number, hereinafter the superscript "unity" refering to the first term in the expansion is omitted.

The boundary conditions are as follows:

$$\psi_1(0) = \psi_1'(0) = f_1(0) = \theta_1(0) = 0, \quad \psi_N(1) = \psi_N'(1) = f_N(1) = \theta_N'(1) = 0. \tag{7}$$

At the interfaces $y = h_i$:

$$\psi_i(h_i) = \psi_{i+1}(h_i) = 0, \quad \psi_i'(h_i) = \nu_{i+1i}\psi_{i+1}'(h_i),$$
$$\theta_i(h_i) = \theta_{i+1}(h_i), \quad \theta_i'(h_i) = k_{i+1i}\theta_{i+1}'(h_i),$$
$$\psi_i''(h_i) - \eta_{i+1i}\nu_{i+1i}\psi_{i+1}''(h_i) = \frac{m_i}{m_1} = \frac{\alpha_{i1}}{\eta_{i1}\nu_{i1}}, \quad f_i(h_i) = 0,$$

where $\alpha_{i1} = \alpha_i/\alpha_1$.

Having solved linear boundary-value problem (6) with conditions (7), we find the eigenvalues λ_i and, correspondingly, the first terms in expansion (5).

The solution to the problem (6)-(7), has the form

$$\psi_i = -\frac{\lambda_i}{6}(y - h_{i-1})(y - h_i)(y - a_i), \tag{8}$$

$$f_i = 2\psi_i' + \frac{\lambda_i}{3} d_i (h_i - a_i),\ i \neq 1,\ f_1 = -\lambda_1 y (3y - 2/3h_1),$$

$$\theta_i = -\frac{\Pr_i \lambda_i}{72} (y - h_{i-1})^2 (3(y - h_{i-1})^2 - 4(y - h_{i-1})(a_i + d_i - h_{i-1}) +$$

$$+ 6d_i (a_i - h_{i-1})) - \sum_{k=1}^{i-1} \frac{\Pr_k \lambda_k}{72} d_k^3 (2(a_k - h_{k-1}) - d_k).$$

Here the designations $h_0 = 0$, $h_N = 1$ are introduced; $a_1 = h_0 = 0$, $a_N = h_N = 1$ for a system consisting of any number of layers. Henceforth a three-layer liquid, N = 3, with mainly be considered. In this case

$$\lambda_1 = \frac{3}{d_1 \gamma} (\alpha_{21} d_{32} - 2(d_{32} + \eta_{32})),$$

$$\lambda_2 = -\frac{3}{d_2 \nu_{21} \gamma} (d_{12}(d_{32} + 2\eta_{32}) + \alpha_{21} d_{32} (d_{12} + 2\eta_{12})), \tag{9}$$

$$\lambda_3 = \frac{3}{d_3 \nu_{31} \gamma} (d_{12} - 2\alpha_{21}(d_{12} + \eta_{12}), \quad a_2 = d_1 \left(1 + \frac{d_{12}}{\nu_{21}} \frac{\lambda_1}{\lambda_2}\right).$$

Here $\gamma = \eta_{21}(d_{12}(d_{32} + 2\eta_{32}) + 2(d_{32} + \eta_{32})(d_{12} + 2\eta_{12}))$, $d_{ik} = d_i/d_k$, i, k = 1, 3.

In the limit $\eta_{32} \to 0$, we obtain from (8), (9) the solutions for a two-layer liquid system with a free surface at the upper boundary, and at $d_{32} \to 0$ for motion of a two-layer liquid in a gap between plates [2]. As in the case of a two-layer liquid with a free surface at the upper boundary, three modes of steady-state motion exist for a three-layer system, depending on the ratios between the parameters of the liquids:

the first

$$\alpha_{21} < \frac{1}{2(1 + \eta_{12}/d_{12})} = \alpha^*;$$

the second

$$\alpha^* < \alpha_{21} < 2(1 + \eta_{32}/d_{32}) = \alpha^0;$$

the third

$$\alpha_{21} > \alpha^0.$$

Figures 2 and 3 show streamlines for different flow modes. For α_{21} pertaining to the first mode, the direction of circulation in the system is determined by the Marangoni force at the interface between the 1st and 2nd liquids, with the most pronounced motion being in the lower layer (Fig. 2a).

The second mode is characterized by competition of the motions initiated by thermocapillary forces at the liquid interfaces. For small α_{21}, one more vortex with an opposite direction of circulation emerges in the 2nd layer near the upper boundary, and the motion in the upper layer is reversed (Fig. 2c). With further increase in α_{21}, the vortex dimensions and intensity increase, a cell at the interface between the 1st and 2nd liquids becomes compressed, the motion in it slows down and, as a consequence, the circulation rate in the lower layer decreases (Fig. 2d, e).

In the third mode, the motion is determined by the Marangoni force at the upper interface of the liquids. The flow pattern is analogous to that in the first mode but with reversed circulation in all vortices (Fig. 3a). With an increase in α_{21}, the intensity of the motion in the 3rd layer (at the chosen values of liquid parameters) grows most quickly. At $\alpha_{21} = \alpha^*$ and $\alpha_{21} = \alpha^0$, the flow modes change. At $\alpha_{21} = \alpha^*$ the upper layer of the liquid is motionless (Fig. 2b), and at $\alpha_{21} = \alpha^0$ the lower layer is at rest (Fig. 2f) in first order in m_1. The flow pattern in two moving layers is the same as in the case of a two-layer liquid moving in a gap between plates. Unlike the motion in a two-layer system with a free surface [2], the ratios of the viscosities and thicknesses of the layers also affect the change in flow modes. For the case of three liquids, one may suppress convection in the upper layer for $\alpha_{21} < 0.5$ by a proper choice of d_{12},

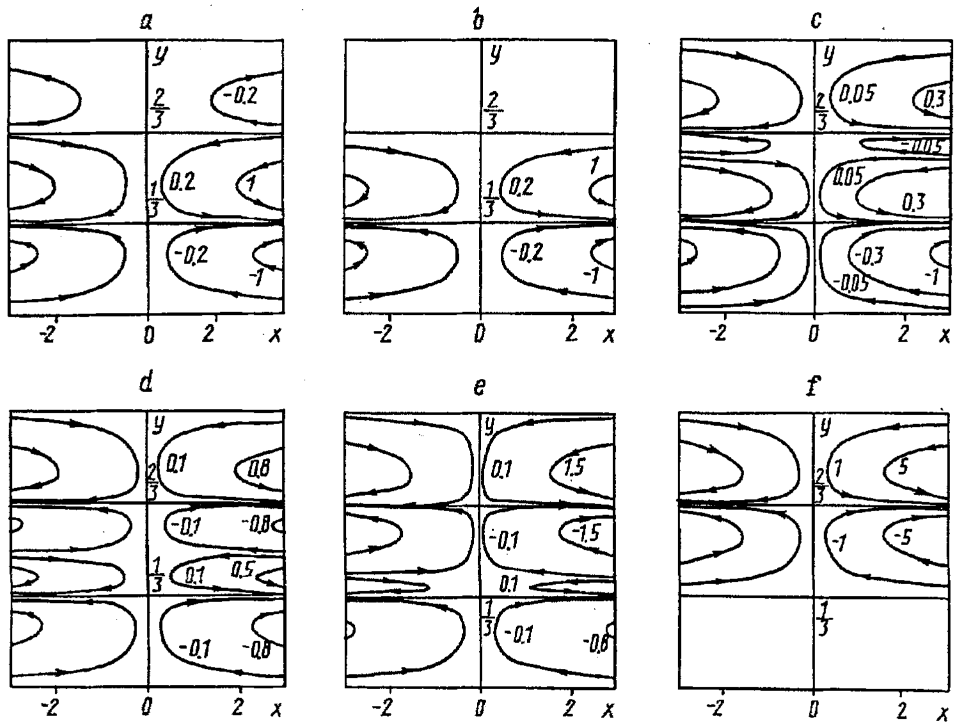

Fig. 2. Streamlines for different modes of the flow ($d_1 = d_2 = d_3 = 1/3$; $\eta_{32} = 0.5$, $\eta_{12} = 3$): a) first mode, $\alpha_{21} = 0$; b) $\alpha_{21} = \alpha^* = 0.125$; c, d, e) second mode, α_{21} = 0.25, 0.5, 1; f) $\alpha_{21} = \alpha^0 = 3$. The streamline values are given in units of $10^{-2} m_1 \nu_1$.

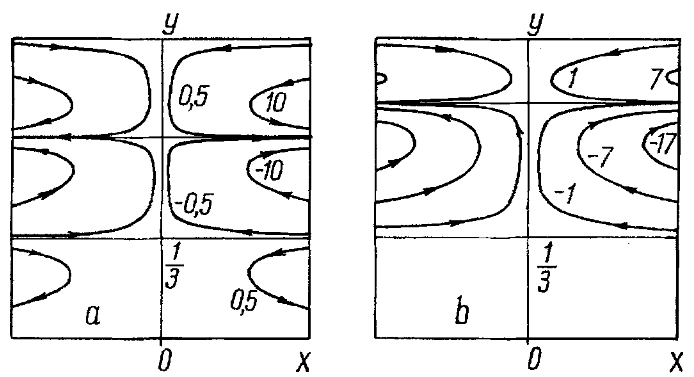

Fig. 3. Streamlines for different thicknesses of the layers ($\eta_{32} = 0.5$; $\eta_{12} = 3$; $\alpha_{21} = 5$): a) third mode, $d_1 = d_2 = d_3 = 1/3$; b) $d_1 = 1/3$, $d_2 = 1/2$, $d_3 = 1/6$; $\alpha_{21} = 5 = \alpha^0$.

and in the lower layer for $\alpha_{21} > 2$ by changing d_{32} (Fig. 3a, b). For $0.5 < \alpha_{21} < 2$ a change in thickness ratios of the layers affects only the position of the interface of vortices in the second layer.

Figure 4 shows the Nusselt number distribution over the depth in units of AHm_1Pr_1 for different ratios of the thermal diffusivities χ_{12} and χ_{13} in the cases $\alpha_{21} = \alpha^*$ (curves 1, 2) and $\alpha_{21} = \alpha^0$ (curves 3, 4). The process of heat transfer by vortices proceeds similarly to the case of a two-layer system with a free surface [2]. We note that at $\alpha_{21} = \alpha^*$ the distribution Nu(y) in the third layer is uniform, and the value of Nu is determined by the value at the interface with the second liquid.

A comparison of curves 1 and 3 (Fig. 4) shows that convection increases the deviation of the temperature from a linear profile. With increase in the thermal diffusivity of the i-th liquid (decrease in χ_{1i}), the effect of the heat conduction is enhanced, and the temperature profile is smoothed (curves 2 and 1, 3 and 4 in Fig. 4).

Since it is found that at $\alpha_{21} = \alpha^*$ the upper layer and at $\alpha_{21} = \alpha^0$ the lower layer are at rest in first order in m_1, it is of interest to elucidate the character of the motion in these layers at higher orders of expansion in m_1.

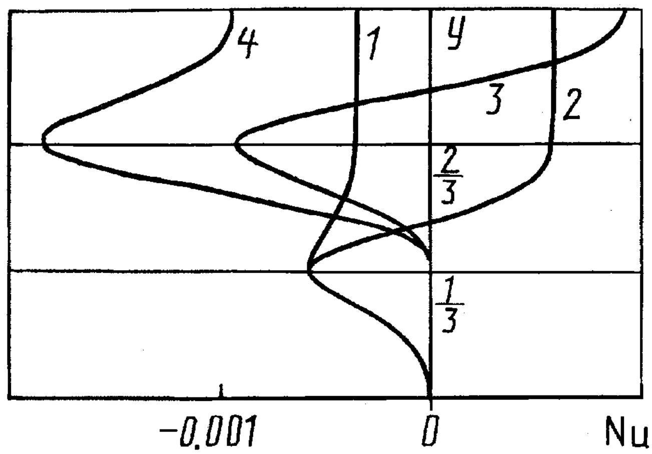

Fig. 4. Nusselt number distribution over the depth in units of AHm_1Pr_1 for $d_1 = d_2 = d_3 = 1/3, \eta_{32} = 0.5, \eta_{12} = 3$: 1) $\alpha_{21} = \alpha^* = 0.125, \chi_{12} = 0.4, \chi_{13} = 0.2$; 2) $\alpha_{21} = \alpha^* = 0.125, \chi_{12} = 2, \chi_{13} = 4$; 3) $\alpha_{21} = \alpha^0 = 3, \chi_{12} = 0.2, \chi_{13} = 0.4$; 4) $\alpha_{21} = \alpha^0 = 3, \chi_{12} = 0.4, \chi_{13} = 0.2$.

For the sake of simplicity, the limiting case of a two-layer system with a free surface ($\eta_{32} = 0$, $\alpha_{21} = \alpha^0 = 2$) was investigated. In second order in m_1, we obtain for the lower liquid

$$\lambda_1^{(2)} = -\frac{d_2^2(\mathrm{Pr}_2 + 47/140)}{d_{12}^2\eta_{21}\nu_{21}(3\eta_{21} + 4d_{21})}, \quad \psi_1^{(2)} = -\frac{\lambda_1^{(2)}}{6}y^2(y - h_1),$$

$$f_1^{(2)} = -\lambda_1^{(2)}y(y - 2/3h_1), \quad \theta_1^{(2)} = -\frac{\mathrm{Pr}_1\lambda_1^{(2)}}{72}y^3(3y - 4h_1) - \tag{10}$$

$$- k_{21}\frac{\mathrm{Pr}_2^2 d_2^4(d_1 + 6)}{336(\eta_{21}\nu_{21})^2}y.$$

The dependences for the stream function and the pressure are of the same form as in the case of the first order in m_1; however, now the velocity field depends on the deviation of the system temperature from a linear profile, and the Prandtl number enters $\lambda_1^{(2)}$. Since the temperature of the lower layer does not change in first order in m_1 (curves 3, 4 in Fig. 4), $\lambda_1^{(2)}$ and, consequently, the velocity do not depend on Pr_1.

In the second order, the temperature of the lower liquid changes over the depth due to both convective heat transfer inside the layer and heat exchange with the second liquid. Since the obtained expressions (10) have no divergent terms, $\psi_1^{(1)} \equiv 0$ at $\alpha_{21} = \alpha^0$, and it follows from (5) that $\psi_1 = m_1^2\psi_1^{(2)} + O(m_1^3)$, $\psi_2 = m_1\psi_2^{(1)} + O(m_1^2)$, etc., then for sufficiently small m_1 the motion in the lower layer is much weaker than that in the upper layer.

At $\alpha_{21} = \alpha^0$, the deviation of the temperature of the second layer from a linear profile is the negative value of the order of m_1 (curves 3, 4 in Fig. 4); therefore, thermocapillary forces on the surface of the second liquid weaken and the intensity of the upper vortex in the second layer at $\alpha_{21} = \alpha^0$ becomes insufficient to suppress the vortex at the interface and the motion in the lower liquid (see Fig. 2e, f). This means that the lower liquid will be motionless at the higher value $\alpha_{21} = \alpha^0 + \varepsilon^0$, where $\varepsilon^0 \sim m_1$, $\varepsilon^0 > 0$. The obtained solution (10) remains correct even at values of α_{21} differing from α^0 by a value of the order of m_1. Then the value of α_{21}, at which no motion is observed in the lower layer, may be evaluated according to (5) as

$$\psi_1 = m_1\psi_1^{(1)} + m_1^2\psi_1^{(2)} + O(m_1^3).$$

Using (8) and (10), we obtain from the condition $\psi_1 \equiv 0$

$$\lambda_1 + m_1\lambda_1^{(2)} + O(m_1^2) \equiv 0.$$

Hence, from (9) at $\eta_{32} = 0$, $\alpha_{21} = 2 + \varepsilon^0$ and (10) we find

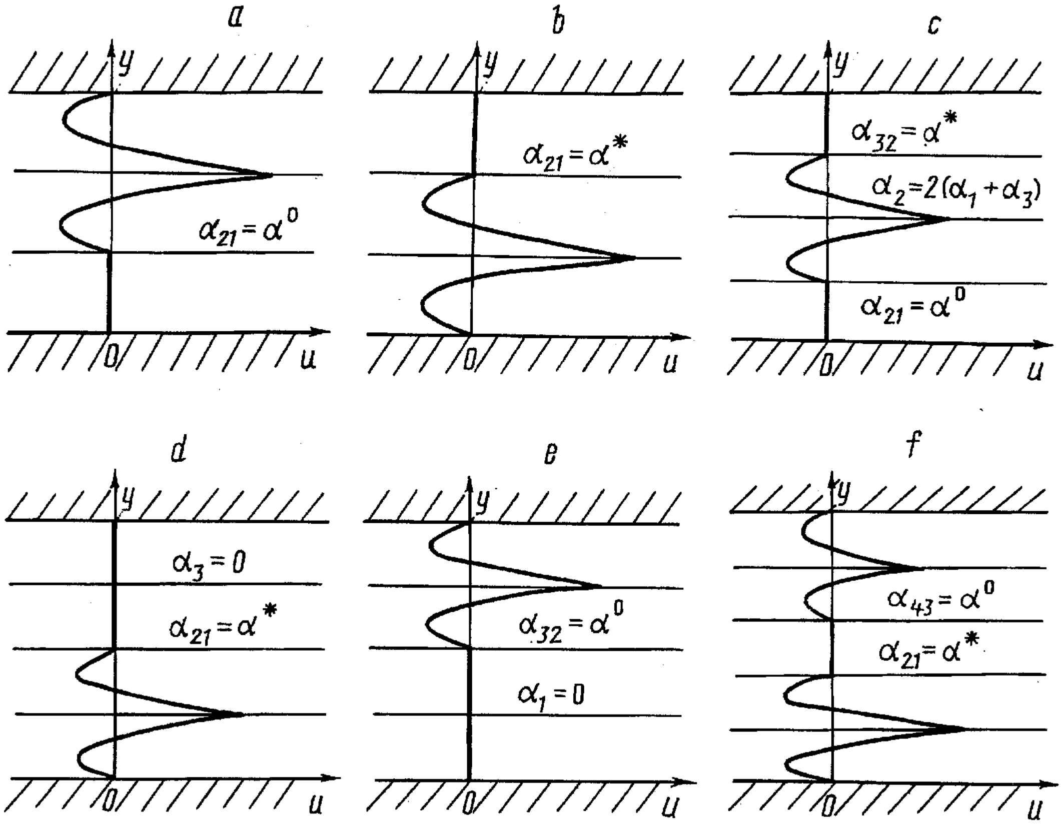

Fig. 5. Schematic of velocity profiles in multilayer liquids: a, b) N = 3; c, d, e) N = 4; f) N = 5.

$$\varepsilon^0 = \frac{m_1 d_2^3 (\mathrm{Pr}_2 + 47/140)}{3\eta_{21}\nu_{21}} + O(m_1^2).$$

Consideration of the case $\alpha_{21} = \alpha^*$ in second order in m_1 yields a similar result, viz., the absence of divergence; the sign of ε^* depends on the ration between Pr_1 and Pr_2.

Analysis of the obtained solutions (8) and the special features of the motion in two- and three-layer liquid systems leads to the following conclusions concerning the motion in multilayer systems with $N \geq 3$.

1. The flow possesses bilateral symmetry relative to the plane $x = 0$. Next, we shall deal with the region $x > 0$.

2. In liquid layers adjacent to solid surfaces only monovortex motion may take place. In inner layers, two vortices with opposite directions of circulation as well as one vortex may exist. The flow pattern in a system with no convection in one of the layers is identical to the situation where the boundaries of a motionless layer are solid surfaces. Therefore, if this motion is to be investigated, one may consider one or several independent systems with a smaller number of layers, taking into account the condition of motionlessness of the i-th layer.

3. In the absence of motion in the i-th layer, for the inner layers from the third to the N-2-th one we may write

$$a_{i-1} = h_{i-1}, \quad \lambda_{i-1} = -\frac{3\alpha_{i-1,1}}{\eta_{i-1,1}\nu_{i-1,1}d_{i-1}}; \tag{11}$$

$$a_{i+1} = h_i, \quad \lambda_{i+1} = -\frac{3\alpha_{i,1}}{\eta_{i+1,1}\nu_{i+1,1}d_{i+1}}. \tag{12}$$

Relation (11) also holds when i = N, and (12) when i = 1. Convective motion in the inner layer when $\alpha_i \neq 0$, $\alpha_{i-1} \neq 0$ may be suppressed when $N \geq 5$.

4. Two neighboring layers may be motionless only in the absence of thermocapillary forces at their interface, $\alpha_i = 0$.

5. The second or N-1-th layer may be at rest only together with the first or N-th layer.

6. If all the Marangoni numbers differ from zero, convection may be suppressed simultaneously in $k \le [(N+2)/3]$ (for $N \ge 3$) layers. In this case a layer at rest is adjacent to systems consisting of two or more moving layers, or one of the boundaries may be a solid surface.

Figure 5 schematically shows velocity profiles in a system consisting of three, four, or five liquid layers in cases where one of the layers is at rest.

NOTATION

H, thickness of the multilayer system; N, number of layers in the system; X, Y, x, y, dimensional and dimensionless coordinates; D_i, d_i, dimensional and dimensionless thickness of the i-th liquid layer; U_i, v_i, horizontal and vertical velocities; T_i, temperature; P_i, pressure; ρ_i, density; $\sigma_i(T_i)$, surface tension coefficient at the i-th interface of the liquids; σ_{0i}, α_i, T_{0i}, coefficients of the temperature dependence $\sigma_i(T_i)$; A, prescribed temperature gradient along the lower solid surface; η_i, ν_i, k_i, χ_i, coefficients of dynamic and kinematic viscosity, thermal conductivity, and thermal diffusivity, respectively; $\eta_{ik} = \eta_i/\eta_k$, $\nu_{ik} = \nu_i/\nu_k$, $k_{ik} = k_i/k_k$, $\chi_{ik} = \chi_i/\chi_k$, $\alpha_{ik} = \alpha_i/\alpha_k$, $d_{ik} = d_i/d_k$, the ratio of the corresponding quantities of the i-th and k-th layers; $Pr_i = \nu_i/\chi_i$, Prandtl number; α^0, α^*, values of α_{ik} at which a change in the modes of stationary liquid motion proceeds. Subscripts: $i = 1$ pertains to the lower liquid; $i = N$ pertains to the upper liquid.

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